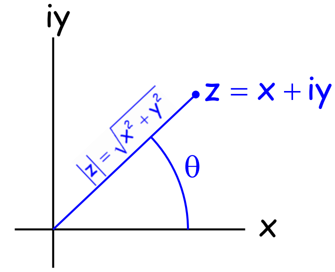
1-D Schrödinger Equation Review

Review Complex Numbers

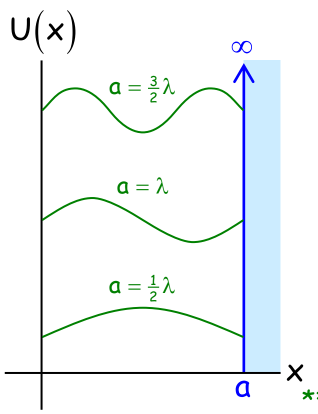


For $z = x + iy$, the complex conjugate is $z^* = x - iy$ and the absolute value is the distance on the complex plane from the origin to the point z.

$$|z|^2 = zz^* = (x+iy)(x-iy) = x^2 + y^2$$

We also utilize Euler's formula stating tha $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$.

Review Chapter 7: The 1-D Schrödinger Equation and Tunneling



A particle in a box: allowed energy states

For a quantum particle of energy, E, in a potential box where U(x) = 0 for 0 < x < a, it can only exist between x = 0 and x = a (since E = K + U and U = ∞ outside of 0 < x < a). **A quantum particle is described by a wave function** that can exist in the box at only certain wavelenths (λ, first three shown) the wave function must be zero at the sides (x = 0 & x = a).

For a standing wave, $\psi(x) = A\sin(kx) + B\cos(kx)$, this requires that

$ka = n\pi$ where n = 1, 2, 3, ... and $k = \frac{2\pi}{\lambda}$ is the wave number

**** Other than saying a quantum particle is described by a wave, this is just math! ****

The physics comes in with de Broglie:

$$p = \frac{h}{\lambda} = \hbar k \quad \text{for} \quad k = \frac{2\pi}{\lambda} \quad \text{and} \quad \hbar = \frac{h}{2\pi} \qquad \text{TZDII}$$

Thus, the allowed momenta of the particle in the box are:

$$a = \frac{n}{2}\lambda = \frac{n\pi}{k} = \frac{n\pi\hbar}{p} \quad \Rightarrow \quad p = \frac{n\pi\hbar}{a}$$

Since the energy of the particle is purely kinetic (U = 0), $E = p^2/2m$ gives the allowed energies:

$$E = n^2 \frac{\pi^2\hbar^2}{2ma^2}, \quad n = 1, 2, 3, \ldots \qquad \text{TZDII (7.23)}$$

The Time-Independent Schrödinger Equation

Physics is about change and its governing equations are differential equations like Newton's Second Law, $\Sigma F = dp/dt$. To find a differential equation for a quantum particle, take derivatives:

$$\psi(x) = A\sin(kx), \quad \frac{d\psi(x)}{dx} = kA\cos(kx), \quad \frac{d^2\psi(x)}{dx^2} = -k^2 A\sin(kx) = -k^2\psi$$

Recalling de Broglie ($p = \hbar k$) and $K = p^2/2m = (\hbar k)^2/2m$ gives

$$\frac{d^2\psi(x)}{dx^2} = -k^2\psi = \frac{2mK}{\hbar^2}\psi$$

where we can replace K = E – U(x) to write the 1-D Schrödinger Equation

$$\frac{d^2\psi(x)}{dx^2} = -k^2\psi = \frac{2m}{\hbar^2}\left[U(x) - E\right]\psi \qquad \text{TZDII (7.39)}$$

Probability Density, Normalization, & Expectation Value

TZDII Equation 6.14 states that $\left|\Psi(\vec{r},t)\right|^2 dV$ is the probability of finding a particle in the volume V at $\vec{r}$ and 6.15 says

$$\left|\Psi(\vec{r},t)\right|^2 = \text{probability (volume) density for finding particle at } \vec{r} \qquad \text{TZDII (6.15)}$$

For our 1-dimensional case, we can write the linear probability density as $\left|\psi(x)\right|^2$ and

$$\text{Prob. of finding particle between } x_1 \ \& \ x_2 = \int_{x_1}^{x_2} \left|\psi(x)\right|^2 dx \approx \left|\psi(x = x_1)\right|^2 \Delta x$$

In general, the probability of finding the particle somewhere is unity, thus, the normalization condition is

$$\int_{-\infty}^{\infty} \left|\psi(x)\right|^2 dx = 1 \qquad \text{TZDII (7.55)}$$

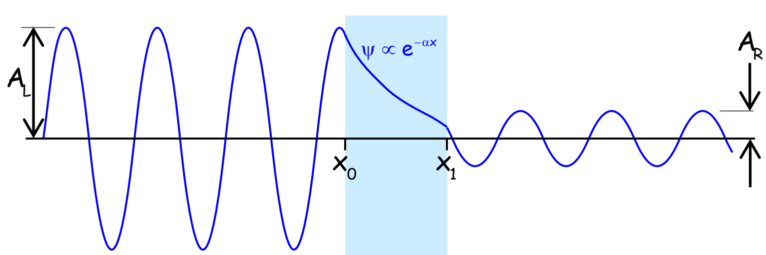
The expectation value (value expected after many measuremnts) of f(x) with a probability density $\left|\psi(x)\right|^2$ is

$$\int f(x)\left|\psi(x)\right|^2 dx = \int f(x)p(x)dx \qquad \text{TZDII (7.69)}$$

Tunneling

Since wave functions can penetrate into regions where $U > E$ as an exponentially decreasing fuction it can penetrate a barrier. If the wave function within the barrier does not go to zero before emerging from it, the wave can emerge on the other side.

The probability density of a wave is proportional to the amplitude squared. Thus the probability that a wave will tunnel through a barrier is the ratio of the squares of the amplitudes of the wave leaving to the wave entering:



$$P_{Tunnel} = \frac{A^2_{Right}}{A^2_{Left}}$$

This ratio is that of the exponential decrease across the barrier, namely that

$$P_{Tunnel} = \frac{e^{-2ax_1}}{e^{-2ax_2}} = e^{-2\alpha(x_1 - x_0)} = e^{-2\alpha L} = e^{\frac{-2L\sqrt{2m(U_0 - E)}}{\hbar}} = e^{\frac{-2L\sqrt{2mc^2(U_0 - E)}}{\hbar c}} \qquad \text{TZDII (7.104)}$$